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The fashion game: Network extension of Matching Pennies



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ABSTRACT

It is impossible, in general, to extend an asymmetric two-player game to networks, because there must be two populations, the row one and the column one, but we do not know how to define inner-population interactions. This is not the case for Matching Pennies, as we can interpret the row player as a conformist, who prefers to coordinate her opponent's action, while the column player can be interpreted as a rebel, who likes to anti-coordinate. Therefore we can naturally define the interaction between two conformists as the coordination game, and that between two rebels as the anti-coordination game. It turns out that the above network extension of Matching Pennies can be used to investigate the phenomenon of fashion, and thus it is named the fashion game. The fashion game possesses an obvious mixed Nash equilibrium, yet we are especially interested in pure Nash equilibrium (PNE for short), whose existence cannot be guaranteed. In this paper, we focus on the PNE testing problem, namely given an instance of the fashion game, answer whether it possesses a PNE or not. Our first result is on the negative side: PNE testing, in general, is hard. For the PNE testing problem restricted to several special structures, i.e. lines, rings, complete graphs and trees, either a simple characterization or an efficient algorithm is provided.

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“Fashion is the most intense expression of the phenomenon of neomania, which has grown ever since the birth of capitalism.”

[Stephen Bayley]

“Fashion has been one of the most influential phenomena in Western civilization since the Renaissance. It has conquered an increasing number of modern man's fields of activity and has become almost ‘second nature’ to us.”

[Lars Svendsen]

1. Introduction

Network games, also known as graphical games, games on graphs and games on networks, roughly speaking, are games that are played on networks. Here the network represents the social connections between players, and payoffs of players are only affected by their neighbors. But action sets of players are usually independent of this social network (notice this difference with routing games on networks). Therefore, any strategic game can be taken as a network game, because we

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can simply let the associated network be complete. Sitting on the intersection of algorithmic game theory, social economics, social physics, social networks (as well as complex networks), and theoretical biology, the field of network games is typically multi-disciplinary. We refer the reader to [16–19,21] for more detailed introduction to this booming field.

A wide class of network games are those that are generalized from two player strategic form games. Theoretically, any two player symmetric game (referred to as the base game) can be generalized to networks in the following way. Each player, denoted by a node of the network, plays the base game once with each of her neighbors. The total payoff of each player is simply defined as the sum of her payoff obtained in all the base games that she is involved. Also, it is usually assumed that for any player, in all the base games that she plays, she should take the same action.

Using a similar logic on an asymmetric two-player game, there must be two populations introduced: the row population and the column population. When two players from different populations meet, they play the original two-player game. Nothing is debatable for this. However, complication comes when two players from the same population meet: we don't know, in general, how to define the game they play. This information is generally not written in the asymmetric two-player game. This is exactly why scholars from evolutionary game theory in the very little literature simply ignore the inner-population game when they deal with asymmetric two-player games. For instance, if the row population stands for deer and column for lions, they only study the relation between deer and lions, but not that among deer or among lions.

In this paper, we shall investigate a network game that is generalized from Matching Pennies, an elementary model in game theory that can be found in any textbook. Note first that Matching Pennies is not symmetric. It turns out that this model is closely related with the phenomenon of fashion.

1.1. The fashion game

Fashion is a very interesting phenomenon that plays a critical role not only in economy but also in many other areas of our society (see [11,23] for more detailed discussions).

On what is fashionable, interestingly, there are two viewpoints that are both extremely popular but almost opposite to each other. One point of view thinks that fashion is a distinctive or peculiar manner or way. Lady Gaga is regarded as fashionable in this sense. The other takes fashion as a prevailing custom or style. Fashion color is fashionable in this latter sense. This difference has a very deep root in psychology (see [11] for more discussions). Following Young [22] and Jackson [18], we shall call the former type of people *rebels*, and the latter *conformists*.

Since fashion comes from comparison with others, and the range that people compare is almost always confined to their friends, relatives, colleagues, and neighbors, that fashion works through a social network is the most natural thing. As far as we know, the model we study is first formulated by Jackson (2008, [18], p. 271), where it is presented as an example of graphical games. It is also studied in Young (2001, [22], p. 38), but the social network is not explicitly expressed there.

Formally, each fashion game is represented by a triple $G = (N, E, T)$, where $N = \{1, 2, \dots, n\}$ is the set of nodes that stand for agents (we may also interchangeably use the term *players*), $E \subseteq N \times N$ the set of edges (no self-loops are allowed), and $T \in \{C, R\}^N$ the configuration of types. For agent $i \in N$, T_i is her type: $T_i = C$ means that i is a conformist, and $T_i = R$ means that she is a rebel. For agents $i, j \in N$, they are neighboring to each other if and only if $ij \in E$. If $ij \in E$, then $ji \in E$, i.e. the network is undirected. The action sets of all agents are identical: $\{0, 1\}$.

We use N_i to denote the set of neighbors of player i , and $\pi_i \in \{0, 1\}$ the action of agent i . Vector $\pi \in \{0, 1\}^N$ is an action profile of G . Given an action profile π , for all $i \in N$, $L_i(\pi) \subseteq N_i$ is the set of neighboring agents that i likes, i.e.

$$L_i(\pi) = \begin{cases} \{j \in N_i : \pi_j = \pi_i\} & \text{if } T_i = C \\ \{j \in N_i : \pi_j \neq \pi_i\} & \text{if } T_i = R \end{cases}$$

Similarly, $D_i(\pi) \subseteq N_i$ is the set of neighboring agents that i dislikes, i.e. $D_i(\pi) = N_i \setminus L_i(\pi)$. Consequently, the utility of each player i is defined as the number of neighbors she likes minus that she dislikes, i.e.

$$u_i(\pi) = |L_i(\pi)| - |D_i(\pi)|.$$

In the rest of this paper, $u_i(\pi)$ may also be referred to as the *satisfaction degree*. An agent is called *satisfied* if $u_i(\pi) \geq 0$. Otherwise, she is called *unsatisfied*. Straightforwardly, an action profile π is a Nash equilibrium if and only if every agent is satisfied.

We may also abuse the term of graph (and network) by calling an instance of the fashion game a graph. For example, when we say that $G = (N, E, T)$ is a tree, we actually mean that the underlying graph (N, E) is a tree. That is, the type configuration T may be simply overlooked.

1.2. Network extension of Matching Pennies

It turns out that, as we have mentioned, the fashion game may also be taken as the network extension of Matching Pennies. First of all, observe that Matching Pennies is equivalent to a very special fashion game: there are two players in total, one is a conformist and the other a rebel, who correspond to the row player and the column player, respectively, and the underlying network is a dyad.

	H	T
H	1, 1	-1, -1
T	-1, -1	1, 1

	H	T
H	1, -1	-1, 1
T	-1, 1	1, -1

	H	T
H	-1, -1	1, 1
T	1, 1	-1, -1

Fig. 1. Three base games. (1) The coordination game (left): conformist V.S. conformist; (2) The anti-coordination game (right): rebel V.S. rebel; (3) Matching Pennies (middle): conformist V.S. rebel.

Let’s see what happens if we try to extend Matching Pennies to networks. As we have argued, there must be two populations, conformists and rebels. The game played between any pair of a conformist and a rebel is clearly Matching Pennies. Compared with general two player asymmetric games, the advantage of Matching Pennies is that the game between two conformists and that between two rebels can both be naturally defined: the (pure) coordination game and the (pure) anti-coordination game. The three base games are presented in Fig. 1.

To be precise, as in the network extension of any symmetric two-player game, we assume in the above extension that (i) each agent plays once with each of her neighbors (one of the three base games according to their types), (ii) for any player, in all the base games that she plays, she should take the same action, (iii) the total utility of each player is simply the sum of the payoffs that she gets from all the base games she plays. Using the above three assumptions and payoff settings in Fig. 1, it can be checked that the fashion game is equivalent to the network extension of Matching Pennies.

Matching Pennies is more basic than the coordination game and the anti-coordination game, because they are both implied in it. Therefore this is reasonable to call the fashion game the network extension of Matching Pennies, although it is also OK to say that the fashion game is the network extension of three base games.

Compared with other elementary models, e.g., Prisoners’ Dilemma, Hawk and Dove, and even Rock–Paper–Scissors, very little attention has been paid to Matching Pennies on networks so far (see Szabo and Fath, [21]). There are mainly two reasons: (i) to most people, no satisfactory physical interpretation of this model is known, (ii) this game is not symmetric. As we have seen, with the interpretation of fashion, the above two barriers are removed now simultaneously.

1.3. Contribution and organization

In this paper, we study the fashion game through the perspective of algorithmic game theory, and focus on the PNE (pure Nash equilibrium) testing problem, i.e. given an instance of the fashion game, answer whether there is at least one PNE. This problem is of interest, because it is directly related with the issue of whether fashion keeps in flux, a key feature of fashion as argued by Young (2001, [22]). When PNE does not exist, this will certainly be true.

Our first result is on the negative side: for the general fashion game, the PNE testing problem is NP-complete, even when we require each node to have an odd degree. This result might indicate that predicting fashion trends is impossible. For several special structures, i.e. lines, rings, complete graphs and trees, where PNE cannot be guaranteed, either a simple characterization (for the first three cases) or an efficient algorithm is provided. Lower bounds for the maximum number of satisfied agents are also provided for the general case as well as for the special cases of lines, rings, and complete graphs.

The rest of this paper is organized as follows. Section 2 provides some warm-up and preliminary results, Section 3 is for the main negative result, Section 4 for the easy special cases, Section 5 for the lower bounds, and Section 6 concludes this paper by pointing out several directions for future research. The subsection below gives a very brief literature review.

1.4. Related work

The exact model of the fashion game has drawn very little attention until now. In fact, to the best of our knowledge, this paper and three subsequent ones of ours [9,11,23] (with other coauthors) are the only serious ones. However, there has been quite a lot of related work, mainly from the fields of social economics and social physics, especially on the network coordination game, one side of the fashion game. We refer the reader to [11,23] for rather complete reviews. This subsection mainly discusses recent developments of network games in the field of algorithmic game theory.

In a subsequent working paper, best response dynamics (both the simultaneous one and the sequential one) are investigated for the fashion game on several special network structures, i.e. lines, rings, and stars, to study fashion cycles [11]. The simultaneous best response dynamic of threshold models on networks, generalizations of the network coordination game, is considered in [1]. For the other side of the fashion game with only rebels, i.e. the network anti-coordination game, there is relatively very little literature. We refer the reader to [4,12] for some tentative results. Computational complexities and approximation algorithms for marketing problems among rebels are discussed in [8]. For more research on social network pricing and marketing problems from the perspective of algorithmic game theory, we refer the reader to [2,6,7,10,15].

Following is a short and incomplete list of remarkable papers on other network game models. Daskalakis and Papadimitriou generalize the min–max theorem of two-player zero-sum games to network games [14]. In Davis et al., price of anarchies of the (symmetric) base game and the corresponding network game is compared [13]. Bindel et al. study the strategic behavior of players in the classical DeGroot model and analyze the price of anarchy [5]. Lu et al. study the influence maximizing problem when diffusion is captured by a linear threshold model [20]. Last but definitely not least, Anshelevich and Hoefer study the price of anarchy for a variant of the Public Good Game [3].

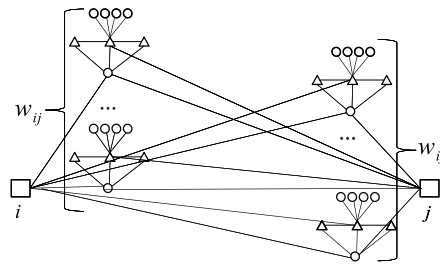


Fig. 2. The simulation.

2. Preliminary results

Recall first that a mixed Nash equilibrium is a mixed action profile such that every player has reached their maximum expected utility, as long as other players do not deviate. Obviously, the fashion game always has a mixed Nash equilibrium: every agent plays 0 and 1 equally likely. Yet we are particularly interested in pure Nash equilibrium (PNE for short). A (pure) strategy profile $\pi^* \in \{0, 1\}^N$ is a PNE if and only if players have no incentive to change their actions. Since $u_i(0, \pi_{-i}) + u_i(1, \pi_{-i}) = 0$ holds for all $\pi_{-i} \in \{0, 1\}^{N \setminus \{i\}}$, π^* is a PNE if and only if

$$u_i(\pi^*) \geq 0, \quad \forall i \in N.$$

Notice that the existence of PNE cannot be guaranteed. To see this, just think about a dyad with one conformist and one rebel, i.e. Matching Pennies. However, as long as PNE exists, they must appear in pairs, because the actions 0 and 1 are symmetric in our setting.

Proposition 1. (See [23].) *The fashion game must have an even number of PNEs.*

Jackson [18] noticed that if each player has no less conformist neighbors than rebel ones, then the existence of PNE is guaranteed. In fact, it can be checked easily that every conformist taking action 1 and every rebel taking 0 is a PNE. This result can be generalized as follows.

Proposition 2. (See [23].) *If the fashion game $G = (N, E, T)$ has one of the following structures, then PNE exists:*

- (a) *For each conformist, she has no less conformist neighbors than rebel ones;*
- (b) *Suppose R is the set of rebels. There exists a partition of R , $\{R_1, R_2\}$ such that $\forall i \in R_1, |N_i \cap R_2| \geq (1/2)|N_i|$ and $\forall j \in R_2, |N_j \cap R_1| \geq (1/2)|N_j|$.*

Remark 1. Due to the above proposition, if either all agents are conformists or all agents are rebels, then PNE definitely exists. In fact, it can be shown easily that in both cases the fashion game possesses a simple potential function (the sum of payoffs, see [23] for more detailed discussions).

On the other hand, we can give a class of structures where definitely PNE does not exist. A natural one is that each conformist is paired with a rebel, and each rebel is paired with a conformist, an extension of Matching Pennies. The following proposition provides a more general class of network structures.

Proposition 3. (See [23].) *If the fashion game $G = (N, E, T)$ is of the following structures, then PNE does not exist: the graph is bipartite, with one side of all rebels and the other side of all conformists, and at least one agent has an odd degree.*

3. PNE testing is hard

The main aim of this section is to show that the PNE testing problem, i.e. given an instance of the fashion game, answer whether it possesses at least one PNE, is hard.

To show this result, we need first to extend the fashion game a little bit by introducing weights on edges: for each edge $ij \in E$, a positive integer w_{ij} is associated and is interpreted as the amount of influence that i and j have on each other. For each agent i , let W_{i1} and W_{i0} be the total weights of her neighbors choosing 1 and 0, respectively. The preference of each conformist i now becomes (the preferences of rebels can be analogously defined):

- If $W_{i1} > W_{i0}$, then she likes 1 more than 0;
- If $W_{i1} < W_{i0}$, then she prefers 0 to 1;
- If $W_{i1} = W_{i0}$, then 1 and 0 are indifferent in her eyes.

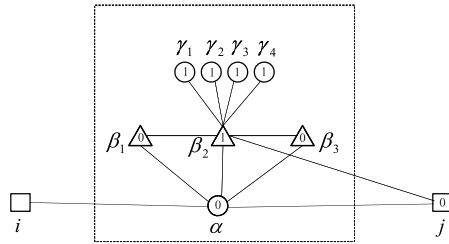


Fig. 3. The gadget.

In the rest of this section, we shall refer to the extended model as WFG (Weighted Fashion Game) and the original unweighted model as FG (Fashion Game). We shall show that for any instance of WFG, there exists an instance of FG such that the PNE testing problems on them have the same answer, and the size of the FG instance is the total weights of the WFG instance times a constant. This tells us that PNE testing on FG is in fact as hard as that on WFG, as long as the total weights are relatively small (polynomially bounded by the number of nodes).

The basic idea is to replace any edge ij of weight w_{ij} with a particular un-weighted graph which can simulate w_{ij} . The simulation is illustrated in Fig. 2, where the two squares represent agents i and j , respectively. We remark that when $w_{ij} = 1$, there is no need for this simulation. In this figure, as in any other figures of the rest of the paper, a square stands for an agent of arbitrary type, a circle for a conformist, and a triangle for a rebel. In this simulation, there are $2w_{ij}$ identical gadgets. For the left w_{ij} gadgets, each of them has one connection to i and two connections to j . For the right w_{ij} ones, each of them has two connections to i and one connection to j . Observe first that the simulation size is indeed a constant times the total weights of the WFG instance. To be more clear and for the convenience of the following proof, we take a left gadget out and name the agents as in Fig. 3.

Lemma 1. *For any instance of WFG, the PNE testing problem has the same answer as that of the simulated FG instance.*

Proof. Denote the instance of WFG as $G = (N, E, T)$, and the simulated FG instance as $G' = (N', E', T')$. We need to show that G has a PNE if and only if G' has a PNE.

Suppose now G has a PNE π . A PNE for G' , denoted as π' , is constructed as follows:

- For each of the old agents (i.e. agents in $N \subseteq N'$, represented as squares in Fig. 2), we assign her the same action as in π ;
- For any edge $ij \in E$, we assume without loss of generality that j is assigned 0 in π (the other case can be done analogously). In π' , we let all the left gadgets have the same action profile as demonstrated in Fig. 3;
- The identical action profile of the right gadgets can be similarly defined.

We show that π' is indeed a PNE of G' .

The new agents (agents in $N' \setminus N$) will not deviate. In fact, in each gadget:

- The four γ s are satisfied, because they are conformists and have an action equal to that of their common and unique neighbor β_2 ;
- β_2 is satisfied, because she has eight neighbors, four of whom choose 1 and the other four choose 0;
- β_1 and β_3 are satisfied, because they both have exactly one 1-neighbor and one 0-neighbor;
- α is satisfied, because no matter what i chooses, as a conformist, she has an action already conforming to the majority of her neighbors (namely β_1, β_3 and j).

The remaining task is to check the old agents (agents in $N \subseteq N'$). Notice that Fig. 2 only depicts a part of i and j 's connections in G' . Two critical observations of Fig. 3 are that: (i) α has the same action as j , and (ii) the actions of α and β_2 are opposite. For agent i , by the above observations, the left gadgets in G' as a whole play a role exactly as j does in G , i.e. they simulate the weight w_{ij} , while the right gadgets do not make any difference to her since they provide the same number of 1-neighbors and 0-neighbors. As i is satisfied in π , we know that she is also satisfied in π' . The satisfaction of j can be similarly demonstrated because the simulation is symmetric to i and j .

To show the sufficiency part, suppose now π' is a PNE for G' . Without loss of generality, we assume again that j chooses 0 in π' . We argue that the action profile given in Fig. 3 is the only feasible one for the new agents. It is enough to focus on β_2 . In fact, β_2 is a rebel and has eight neighbors, four of whom, the γ s, are quite annoying to her: they are conformists, have β_2 as the only neighbor, and therefore will always choose whatever β_2 chooses. Being satisfied, the only possibility for β_2 is to have all the other neighbors assigned an action different with her. Since the action of j is 0, we have immediately the action profile in Fig. 3, which completes the whole proof. \square

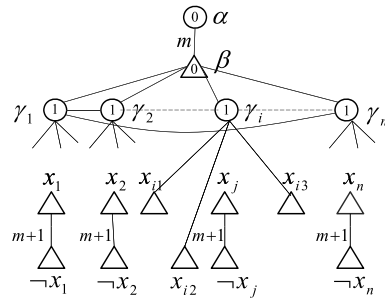


Fig. 4. The reduction.

Lemma 2. *The PNE testing problem for WFG is NP-complete.*

Proof. First of all, it is obvious that this problem belongs to NP. To show its NP-completeness, we do reduction from the 3SAT problem.

For any instance of the 3SAT problem with n variables x_1, x_2, \dots, x_n , and m clauses $x_{i1} \vee x_{i2} \vee x_{i3}, 1 \leq i \leq m$, where $x_{ij} \in \{x_1, x_2, \dots, x_n, \neg x_1, \neg x_2, \dots, \neg x_n\}, 1 \leq i \leq m, 1 \leq j \leq 3$, construct an instance of the PNE testing problem of WFG as follows. There are four levels of agents (each agent can only have neighbors from the same and adjacent levels):

- The first level consists of only one agent α , who is a conformist;
- The second level has also only one agent, a rebel β ;
- For the third level, the m agents $\gamma_1, \gamma_2, \dots, \gamma_m$ are all conformists, and each of them corresponds to a clause in the 3SAT instance;
- The fourth level consists of $2n$ rebels, each of whom corresponds to either a variable or the negation of a variable. We denote them still as x_i and $\neg x_i$.

The network structure and weights of the constructed WFG instance are as follows:

- There is one link between α and β with weight m ;
- β is linked to all the agents in the third level with unit weights;
- The third level agents form a unit-weight-cycle, i.e. γ_1 is connected to γ_2 and γ_m, γ_2 to γ_1 and γ_3 , and so on, and all the weights are 1;
- For each γ_i corresponding to clause $x_{i1} \vee x_{i2} \vee x_{i3}$, she is also connected to the fourth level agents x_{i1}, x_{i2} and x_{i3} with unit weights;
- In the fourth level, each of the variable agent is paired with the corresponding negation variable agent with weight $m + 1$.

Obviously, this construction can be done in polynomial time. The reduction is illustrated in Fig. 4. Notice that the unit weights are not expressed.

We show in the rest of this proof that the answer to the 3SAT instance is “YES” if and only if the WFG instance has a PNE.

To show the necessity part, suppose now there is an assignment π to the 3SAT instance such that all the clauses are true. We construct a PNE π' as follows (illustrated partially in Fig. 4):

- α and β both choose 0;
- All the third level agents choose 1;
- For each of the fourth level agents, she is assigned action 1 if and only if the corresponding variable is assigned truth in π .

We check below that the above constructed π' is indeed a PNE:

- Obviously, α and β will not deviate;
- For each γ_i in the third level, she is a conformist and has six neighbors. Since her action conforms to two neighbors in the same level, and conforms to at least one neighbor in the fourth level, she is satisfied;
- For agents in the fourth level, notice that they may have many but at most m neighbors from the third level. Since each agent has exactly one neighbor from the same level, who chooses a different action, and the weight is big enough, we know that the agents in the fourth level are all satisfied, no matter whatever the third level agents choose.

To show the sufficiency part, suppose now the WFG instance has a PNE π' . Without loss of generality, suppose α chooses 0 in π' . Construct an assignment π of the 3SAT instance naturally by letting any variable choose truth if and only if the corresponding agent in the fourth level chooses 1. We need to show that each clause is true under this assignment.

First of all, it is clear in π' that each agent corresponding to the negation of a variable in the fourth level chooses 1 if and only if the paired agent corresponding to that variable chooses 0, because the weight of the edge between them is quite large. Since α chooses 0, and she is a conformist with only one neighbor β , we know that β chooses 0 in π' . β , as a rebel, has a neighbor α with a large weight choosing the same action as her. Hence it must hold that the γ agents in the third level all choose action 1. For γ_i , a conformist choosing action 1, she has two 1-neighbors in the third level, and one 0-neighbor in the second level. To be satisfied, it must hold that at least one of her neighbors in the fourth level chooses 1, which proves that the corresponding clause is true under this assignment and completes the whole proof. \square

Combining [Lemma 1](#) and [Lemma 2](#), we have the following main result.

Theorem 1. *The PNE testing problem is NP-complete for the fashion game.*

In the field of social physics, it is usually assumed that every agent has an odd number of neighbors such that no tie is involved. We show in the rest of this section that for this special case, it is still NP-complete to test whether PNE exists. The following definitions are convenient for the presentation of our result and its proof.

Definition 1. We say that a network is odd if and only if each of its nodes has an odd degree. We further abuse the term by calling a fashion game odd if the underlying network is odd.

Definition 2. Let G be a fashion game. $O(G)$, the associated odd fashion game of G , is defined by growing from G as follows: for each agent in G that has an even degree (if any), we add a pendant agent who links only to the given agent and is of the same type.

Clearly, $O(G)$ is always odd. And when G itself is odd, we have $O(G) = G$. The following lemma is critical.

Lemma 3. *G possesses a PNE if and only if $O(G)$ possesses a PNE.*

Proof. Suppose G possesses a PNE π . We define $O(\pi)$, an action profile of $O(G)$, simply by letting the old agents choose the same actions as in π , and the new pendant ones choose whatever they like. The fact that the pendant agents are of the same type as their unique neighbors implies that $O(\pi)$ is indeed a PNE of $O(G)$. To show the opposite side, suppose now $O(G)$ has a PNE $O(\pi)$. We define an action profile π of G simply by letting all the agents choose the same actions as in $O(G)$. For any agent i in G , who is also in $O(G)$, $O(\pi)$ is a PNE of $O(G)$ implies that the utility of i in $O(G)$ is nonnegative. Since every agent has an odd degree in $O(G)$, we know that the utility of i in $O(\pi)$ is at least 1. Combining with the fact that the utility of any agent in π is at least her utility in $O(\pi)$ minus 1, we arrive at that i is satisfied in π and complete the proof. \square

Theorem 2. *The PNE testing problem is NP-complete for the odd fashion game.*

Due to [Lemma 3](#), correctness of the above theorem is straightforward, because it can be proved by the natural reduction below from the PNE testing problem of the fashion game, which has been proved to be NP-hard in [Theorem 1](#): given any instance G of the PNE testing problem for the fashion game, we reduce it to its associated odd fashion game $O(G)$.

4. Easy cases

Since PNE testing is hard for the general situation, it is meaningful to consider special cases. In this section, we shall consider lines, rings, complete graphs and trees. We demonstrate that PNE testing is easy for each of them, and there are simple characterizations for the first three cases.

4.1. Lines

To characterize the line case, the following observation is critical.

Lemma 4. *Suppose that G is a line and G' a subline of it (with at least two nodes). If G' possesses a PNE, then so does G .*

Proof. We can grow G from G' via adding new nodes one by one. In each step, PNE can be guaranteed, because we can simply let the old agents choose the same actions as in the last step, and let the new agent choose an action that she

likes. This may go wrong only on the new agent's neighbor. However, her utility is one in the last step, and thus can still guarantee nonnegative in the current step. \square

Note that in-existence of PNE for a subline does not imply that the whole line has no PNE either. To see this, let's check the following example.

Example 1. G is a line with at least three agents, who, except the left-most one, are all conformists. Then the subline composed of the left-most two agents has no PNE, because it is equivalent to Matching Pennies. However, the entire instance does possess at least one PNE. To see this, we can simply let the left-most rebel choose 0 and all the others choose 1.

Theorem 3. *If G is a line, then PNE does not exist if and only if conformists and rebels are alternating on it.*

Proof. Note first that for any dyad, it has a PNE if and only if the two agents are of the same type. Therefore, given any line, we know by Lemma 4 that as long as there are two adjacent agents of the same type, the line must possess a PNE. This proves the necessity part.

We are left to show the sufficiency part. Suppose now G is a line with alternating conformists and rebels. Then the total payoff of all agents is zero for any action profile, i.e.

$$\sum_{i \in N} u_i(\pi) = 0, \quad \forall \pi \in \{0, 1\}^N. \tag{1}$$

Eq. (1) is valid because each edge always contributes zero to the total payoff: it is always liked by one agent and disliked by the other, no matter what action profiles the corresponding two agents choose.

Suppose now on the contrary that G has a PNE π^* , which implies that

$$u_i(\pi^*) \geq 0, \quad \forall i \in N. \tag{2}$$

Combining (1) and (2), we have $u_i(\pi^*) = 0, \forall i \in N$. However, this cannot be true for the two endpoint agents, because they have both only one neighbor. We get a contradiction and complete the proof. \square

4.2. Rings

Theorem 3 also implies that only in very rare cases, we cannot guarantee PNE existence on lines. For rings, it is even rarer.

Theorem 4. *If G is a ring, then PNE does not exist if and only if conformists and rebels are alternating and $n/2$ is odd.*

Proof. Suppose G is a ring with alternating conformists and rebels. This implies that any action profile of G has a total satisfaction degree of exactly 0 (recall (1) in the proof to Theorem 3). Therefore, in any PNE, the satisfaction degree of each agent is exactly 0. Hence in any PNE, for each agent, it must be true that one of her neighbors chooses 1 and the other chooses 0. This can only happen when two 0s alternate with two 1s on the ring, which is impossible when $n/2$ is odd.

To complete the proof, we only need to show that when conformists and rebels are not alternating on the ring, PNE always exists. First of all, if all agents are conformists or all agents are rebels, we are done (recall Remark 1). So suppose now there is at least one conformist and at least one rebel. Then there must exist a CR edge. We cut the ring from any CR edge into a line, then by Theorem 3 PNE exists on the line. Suppose π is a PNE for this line, then it is also a PNE for the ring, because putting the cut edge back can reduce the satisfaction degrees of the endpoint agents by at most 1 and the satisfaction degrees of the endpoint agents on the line are both 1 in π . \square

4.3. Complete graphs

Fix an instance of the fashion game G . For any agents i and j , recall that N_i and N_j are their neighbor sets. For any two agents i and j , we call $C(i, j) = N_i \cap N_j$ their common set of neighbors. For any action profile π , we let $\#_{ij}1$ be the number of agents choosing 1 in $C(i, j)$, and $\#_{ij}0$ be that of choosing 0. The following definition is also useful.

Definition 3. If $N_i \setminus \{j\} = N_j \setminus \{i\}$, we say that i and j are symmetric.

Note that for complete graphs, any pair of agents are symmetric.

Lemma 5. *Suppose i and j are symmetric and connected. If they are both satisfied in some action profile π , then the following statements are true.*

- (a) When both agents are conformists, they must choose the same action in π .
- (b) When one agent is a conformist, the other is a rebel, and they choose the same action, it holds that $|\#_{ij}1 - \#_{ij}0| = 1$. Hence $\#_{ij}1 + \#_{ij}0$ is odd.
- (c) When both agents are rebels and they choose different actions, it holds that $|\#_{ij}1 - \#_{ij}0| \leq 1$.

Proof. (a) Suppose on the contrary that i chooses 1 and j chooses 0. That i is satisfied implies that $\#_{ij}1 \geq \#_{ij}0 + 1$, and j is satisfied means that $\#_{ij}0 \geq \#_{ij}1 + 1$. The two inequalities contradict each other.

(b) Suppose without loss of generality that i is a conformist, j is a rebel, and they both choose 1. That i is satisfied implies that $\#_{ij}1 + 1 \geq \#_{ij}0$, and j is satisfied means that $\#_{ij}0 \geq \#_{ij}1 + 1$. Combining the two inequalities we have $|\#_{ij}1 - \#_{ij}0| = 1$. The oddness of $\#_{ij}1 + \#_{ij}0$ is immediate.

(c) Suppose i and j choose 1 and 0, respectively. Then by satisfaction of them we have $\#_{ij}0 + 1 \geq \#_{ij}1$ and $\#_{ij}1 + 1 \geq \#_{ij}0$, which give the result immediately. \square

Notice that in the above lemma, the symmetry concept could be weakened. Below is the main result of this subsection.

Theorem 5. Suppose G is a complete graph. Let c and r be the number of conformists and rebels, respectively. Then PNE exists on G if and only if one of the three conditions hold:

- (a) $c \geq r + 1$;
 (b) $0 < c \leq r$ and n is odd;
 (c) $c = 0$.

Proof. The sufficiency part can be verified easily as follows.

- When $c \geq r + 1$, let all the conformists choose 1 and rebels choose 0;
- When $0 < c \leq r$ and n is odd, let $(n - 1)/2$ rebels choose 0 and all the other agents choose 1;
- When $c = 0$, the existence of PNE is immediate from the general positive result for the case with all rebels (recall Remark 1).

For the necessity part, we need to consider the only case left: $0 < c \leq r$ and n is even. Suppose on the contrary that PNE exists. By Lemma 5(a), all the conformists must choose the same action, say 1. Condition $0 < c \leq r$ tells us that at least one rebel should choose 1, because otherwise the conformists cannot be satisfied. For any conformist and any rebel taking 1, using Lemma 5(b), we know that n is odd, contradicting the hypothesis. \square

4.4. Trees

For trees, there is no such simple characterization as for lines, rings, or complete graphs. But still, the existence of PNE can be tested efficiently. The main idea is to reduce the PNE testing problem of any tree into that of a smaller tree by dropping the leaf nodes gradually, and modifying the types of the corresponding parents of leaf nodes (when necessary).

Since the behaviors of leaf agents are simple, the following two observations are obvious.

Lemma 6. If G is a tree, then for any parent of leaf nodes:

- (a) Dropping the same number of her children from both types doesn't change the answer to the PNE testing problem.
 (b) If she has less children of her own type than children of the different type, and the gap is at least two, then G has no PNE.

The following simple property from [11] is also useful.

Proposition 4. (See [11].) If G is a star, then PNE exists for G if and only if at least half of the peripheral agents are of the same type as the central agent.

Now we can describe the algorithm.

The Shrinking Algorithm

Input the tree. For any parent of leaf nodes, there are three cases:

1. If she has less children of her own type than children of the different type, and the gap is at least two, then output "NO" and stop;

2. If she has identical number of conformist children and rebel ones, delete all her children;
3. In all the other cases (that is, either she has more children of her own type than children of the different type, or she has exactly one less child from her own type than children from the different type), let t_1 be the type of her parent, t_2 be her own type, t_3 be the type of her children that has more members than the other type, $t_1, t_2, t_3 \in \{C, R\}$. Then:
 - 3.1 when $t_1 = t_3$, delete all her children;
 - 3.2 when $t_1 \neq t_3$, delete all her children, and change her type to t_1 (if $t_2 \neq t_1$).
4. Repeat the above process for the new tree until either it stops in step 1 or the tree shrinks to a star. Suppose it finally shrinks to a star, then output “NO” if the central agent has less children of her own type than children of the different type, and “YES” otherwise.

Theorem 6. *If G is a tree, then the Shrinking Algorithm checks whether it has a PNE in linear time.*

Proof. We only need to show that when the algorithm goes to step 3, the answer to the PNE testing problem of the new tree does not change. Denote the tree before that step as F , and the tree after that step as F' . Call the parent of leaf nodes under consideration i . We discuss in three cases.

Case 1. $t_1 = t_2$. According to the algorithm, the only difference between F and F' is that some leaves in F are deleted. Notice that in F either i has more children of her own type than that of the different type, or she has exactly one less child of her own type than that of the different type. In either case, if the answer to F' is “YES”, so is the answer to F . Suppose now the answer to F is “YES” and π is a corresponding PNE. A PNE for F' , denoted as π' , can be constructed naturally as follows: (i) agents in F' , except i , all choose the same action as in F , (ii) i chooses the same action as her parent if she is a conformist, and an opposite action if she is a rebel. Since $t_1 = t_2$, we know that in whichever case, the action chosen by i is also liked by her parent, which will make things no worse for her parent. Since the only agent that may possibly become unsatisfied is i 's parent, we know that π' is a “YES” certificate for F' .

Case 2. $t_1 = t_3 \neq t_2$. Again, according to the algorithm, the only difference between F and F' is that some leaves in F are deleted. $t_3 \neq t_2$ means that i has exactly one less child of her own type than those of the different type. Therefore, if the answer to F' is “YES”, so is the answer to F . Suppose now the answer to F is “YES” and π is a certificate. Agent i must have chosen an action in π that is not liked by her parent but by herself. Hence simply letting all the agents in F' choose the same actions as in F will give a PNE.

Case 3. $t_2 = t_3 \neq t_1$. According to the algorithm, F' is derived by deleting the children of i and changing her type to t_1 . Suppose now the answer to F' is “YES”, and let π' be a PNE for it. We construct π , a PNE for F , simply by letting all the children of i choose actions they love and all the other agents choose the same action as in F' . $t_2 = t_3$ means that i has more children of her own type than that of the different type in F , so she is satisfied in π , and hence π is a PNE for F . To show the other side, suppose now the answer to F is “YES” and π is a PNE of F . Simply letting i choose an action she likes and all the other agents choose the same actions as in π will give a PNE for F' , because i and her parent are of the same type as in F' . \square

From proof to the decision version [Theorem 6](#), it can be observed that the following stronger solution version is also valid.

Theorem 7. *If G is a tree and PNE exists for it, then we can find one in linear time.*

5. Optimization problems and lower bounds

As a corollary of [Theorem 1](#), it is trivial that maximizing the total number of satisfied agents is NP-hard. Another natural optimization problem is also hard.

Theorem 8. *For the fashion game, maximizing the total amount of satisfaction degree is NP-hard.*

Proof. There are three types of edges in total, CC, CR, and RR. For any action profile, a CR edge always contributes zero to the total amount of satisfaction degree, because either this relation is liked by the corresponding conformist but disliked by the rebel, or it is liked by the rebel but disliked by the conformist. Therefore, we can safely assign to all the conformists the same action, and it is without loss of generality for us to consider a graph where all the agents are rebels. The optimization problem for this special case is equivalent to the MaxCut problem, which is NP-hard. \square

The following example shows that when all agents are satisfied, the total amount of satisfaction degree may not be maximized.

Example 2. There are 4 agents in total, who are all conformists and form a cycle. Then the action profile where two adjacent agents choose 1 and the other two choose 0 is a PNE, because each agent has a utility of exactly zero. Hence the

total amount of satisfaction degree in this PNE is also zero. However, if we let all agents choose the same action, then the total payoff will be 8.

Let π be an action profile of G . We use $s(\pi)$ to denote the number of satisfied agents in π . Then

$$\text{opt}(G) = \max\{s(\pi) : \pi \text{ is an action profile of } G\}$$

is the maximum number of satisfied agents. We first provide a lower bound of $\text{opt}(G)$ for the general case.

Theorem 9. *In the fashion game, $\text{opt}(G) \geq n/2$. The bound is tight only in the case that G has exactly $n/2$ components and each component is a conformist–rebel dyad.*

Proof. We prove through a simple probabilistic method. Let each agent play 1 and 0 equally likely as in the mixed equilibrium. Then for each agent, the probability that she is satisfied is at least 0.5, and the expected number of satisfied agents is no less than $n/2$. Therefore, there exists at least one action profile where at least $n/2$ agents are satisfied. Hence the former part of this theorem is valid.

If G has exactly $n/2$ components and each component is a conformist–rebel dyad, the bound is obviously tight. We only need to show the other side. Suppose now $\text{opt}(G) = n/2$. Then, due to the probabilistic argument, for any action profile of G , there are exactly $n/2$ satisfied agents, i.e.

$$s(\pi) = n/2, \quad \forall \text{ action profile } \pi \text{ of } G. \quad (3)$$

Eq. (3) is true because otherwise we would have $\text{opt}(G) > n/2$, contradicting the hypothesis that $\text{opt}(G) = n/2$. Let e be the action profile of G where all agents choose 1. Then in e all conformists are satisfied, and all rebels are unsatisfied. Therefore, (3) implies immediately the following claim.

Claim (i). *G has exactly $n/2$ conformists and $n/2$ rebels.*

Take an arbitrary rebel i , let her action change to 0, and denote the new action profile as e' .

Claim (ii). *Rebel i has at least one conformist neighbor.*

Claim (ii) can be proved as follows. Suppose on the contrary that all neighbors of rebel i are conformists. Then all her neighbors would have no less satisfaction degree in e' than in e . Combining with the fact that agent i is satisfied in e' and unsatisfied in e , we get $s(e') \geq s(e) + 1 = n/2 + 1$, contradicting (3).

Claim (iii). *At least one of rebel i 's conformist neighbors has degree one.*

Claim (iii) can be proved as follows. Notice that, besides i , only the agents who are i 's neighbors and have degree one have different satisfaction states: her rebel leaf neighbors are unsatisfied in e but satisfied in e' , and her conformist leaf neighbors are satisfied in e but unsatisfied in e' . Let c_i be the number of i 's conformist leaf neighbors, and r_i be the number of her rebel leaf neighbors, then

$$s(e') = s(e) + r_i - c_i + 1.$$

Eq. (3) implies that $s(e) = s(e')$ and hence it must be true that $c_i \geq 1$, which proves Claim (iii).

Combining Claim (i) and Claim (iii), it can only be that each rebel has exactly one conformist neighbor, whose degree is one. Finally observe that no pair of rebels could be connected, because otherwise we can easily construct an action profile where at least one of the rebels is satisfied while still keeping all the conformists being satisfied (e.g., let all agents, but the rebel who has at least one rebel neighbor, choose 1). This completes the proof. \square

Remark 2. Using an argument similar to the proof of Proposition 2(a), it can be shown that there exists an action profile such that either all the conformists are satisfied, or all the rebels are satisfied. This can serve as a new proof to the former part of Theorem 8. See [23] for detailed discussions.

We have better lower bounds on special structures.

Theorem 10. *Let G be an instance of the fashion game.*

- (a) *If G is a line, then $\text{opt}(G) \geq n - 1$;*
- (b) *If G is a ring, then $\text{opt}(G) \geq n - 1$;*
- (c) *If G is a complete graph, then $\text{opt}(G) \geq 3n/4$.*

Proof. By discussions in the proofs to [Theorem 3](#) and [Theorem 4](#), (a) and (b) are straightforward. We are left to show (c).

Suppose G is a complete graph, we still use c to denote the number of conformists and use r to denote the number of rebels. By [Theorem 5](#), we assume that $0 < c \leq r$ and $n = c + r$ is even. We want to show that:

Claim. $opt(G) = r$ when $c \leq n/4$ and $opt(G) = n/2 + c - 1$ when $c > n/4$.

Case 1. When $c \leq n/4$, let all the conformists and $n/2 - c$ rebels choose 1, and all the other rebels choose 0, it can be observed that all the rebels are satisfied, and none of the conformists is satisfied, therefore $opt(G) \geq r$. This is the best we can do. In fact, suppose there are at least $r + 1$ satisfied agents, then there must exist one satisfied conformist i . Say i chooses 1. That i is satisfied implies that there are at least $n/2 + 1$ agents in total choosing 1, and at least $n/2 + 1 - c \geq n/4 + 1$ of them are rebels, who are not satisfied. Since $c \leq n/4$, we have $n - (n/4 + 1) < r + 1$, a contradiction.

Case 2. When $c > n/4$, let all the conformists and $n/2 - c + 1$ rebels choose 1, and all the other rebels choose 0. It can be observed that all the conformists, and the rebels taking action 0 are satisfied, and none of the rebels taking action 1 is satisfied. Since $c + (n/2 - 1) \geq 3n/4$, we get $opt(G) \geq 3n/4$. We show next that this is the best we can do. First of all, $c + (n/2 - 1) > r$. Suppose there are at least $c + n/2$ satisfied agents, then there must exist one satisfied conformist i . Say i chooses 1. That i is satisfied implies that there are at least $n/2 + 1$ agents in total choosing 1, and at least $n/2 + 1 - c$ of them are rebels, who are not satisfied. Since $(n/2 + 1 - c) + (c + n/2) > n$, we get a contradiction and complete the proof. \square

The lower bounds for lines and rings are both tight, because there exist examples where PNE cannot be guaranteed. From the claim in the above proof, it can be seen that the lower bound for complete graphs is also tight. To end this section, let's consider a problem about monotonicity of $opt(G)$.

Definition 4. Let G and G' be two instances of the fashion game. If the underlying graph of G' is an induced subgraph of that of G and the agent types are consistent, we say that G' is a subgame of G .

Suppose G' is a subgame of G , then is it always true that $opt(G') \leq opt(G)$? The following example shows very interestingly that the answer is “NO”.

Example 3. Let G' be a complete graph with 1 conformist and 4 rebels, and G be a complete graph with 2 conformists and 4 rebels. Then G' is a subgame of G . Since the number of nodes of G' is odd, we know from [Theorem 5](#) that G' possesses at least one PNE, i.e. $opt(G') = 5$. From the proof to [Theorem 10\(c\)](#) (to be precise, the claim), however, we know that $opt(G) = 4$.

6. Concluding remarks

In this paper, we investigated the fashion game, i.e. the network extension of Matching Pennies, through the perspective of algorithmic game theory. We focused on the PNE testing problem as well as the lower bounds for the maximum number of satisfied agents.

Many open problems are still waiting to be explored. We list some of them in which the authors are interested most.

- (1) Consider the PNE testing problems for more special cases, say the bipartite graph, planar graphs, regular graphs, etc.
- (2) Since $n/2$, the lower bound of the maximum number of satisfied agents, is tight only for a rather special graph that is disconnected, can we improve this bound for connected graphs? We conjecture that the tight bound for connected graphs is $2n/3$.
- (3) Design approximation algorithms for various optimization problems with different objective functions. There are at least five natural objective functions. (i) The number of satisfied agents; (ii) The total amount of satisfaction degree. Notice that the optimal value of this function is always nonnegative (using a similar argument as in the proof to [Theorem 8](#)). Most importantly, the optimal value is zero if and only if the graph is a bipartite graph with one side of all rebels and one side all conformists, in which case any configuration has an objective value of zero. Therefore, approximation problem for this objective value is meaningful; (iii) The total amount of *satisfaction level*, where the satisfaction level of each agent is defined as the number of neighbors she likes. An agent i is satisfied if and only if her satisfaction level is equal to or greater than $|N_i|/2$; (iv) The total amount of *normalized satisfaction degree*, where the normalized satisfaction degree of agent i is defined as her satisfaction degree divided by her degree; (v) The total amount of *normalized satisfaction level*, where the normalized satisfaction level of agent i is defined as her satisfaction level divided by her degree.

(4) Discuss weighted versions of the fashion game. Notice that weights can be assigned to edges, as we do in the proof to [Theorem 1](#), and they can also be assigned to nodes.

(5) Consider directed networks, i.e. graphs where $j \in N_i$ does not always imply $i \in N_j$. This model is more realistic than our original one.

(6) Consider a generalization of the fashion game with more than two actions.

(7) Models with continuous types of agents. Rebel and conformist are just two extreme types of agents. In reality, most people behave in between. For example, I do not want to be the first guy to wear a very strange T-shirt, but if the proportion

of people around me who wear it reaches a certain threshold, then I can accept it. However, if all the people around me are wearing it, I will not wear it either. Generally, the type of each person can be represented by an interval $[l, u]$, $0 \leq l < u \leq 1$, which means that an action that is taken by a proportion of people that falls into this interval is acceptable. A rebel is of type $[0, 1/2]$ and conformist of type $[1/2, 1]$.

(8) Stubborn agents. In the current model of the fashion game, the action 0 or 1 has no absolute value to any one. The color of my sweater, blue or red, means nothing to me. The only thing that matters is the comparison with others. This is indeed the essence of fashion. But in reality, many people have their private values over different actions. We call the ones who are not fond of fashion at all stubborn agents. The effect of stubborn agents is an interesting topic.

(9) Random graph models, say ER random graph, small-world networks, and scale-free networks.

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